University College London
DEPARTMENT OF MATHEMATICS
Mid-Sessional Examinations 2012
Mathematics 1101
Wednesday 11 January 2012 2:30 – 4:30

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination*

1. (a) State the definition of convergence for a sequence of real numbers.

   (b) Using only the definition of convergence, prove that $\lim_{n\to\infty} \frac{3n+1}{2n+5} = \frac{3}{2}$.

   (c) Prove that if $\langle x_n \rangle$ and $\langle y_n \rangle$ are sequences with $\lim_{n\to\infty} x_n = \ell$ and $\lim_{n\to\infty} y_n = m$, then $\lim_{n\to\infty}(x_n + y_n) = \ell + m$.

   (d) Using any techniques at your disposal, compute the limit of the sequence
$$x_n = \frac{2^n + 3^n}{2^n + (-4)^n}.$$

2. (a) Find, if they exist, the sup, inf, max and min of the set
$$S = \{x \in \mathbb{R} \mid x^2 - 2x < 0 \text{ and } x \geq 1\},$$
and briefly explain your answers.

   (b) Consider the sequence $\langle x_n \rangle$ defined recursively by
$$x_1 = \frac{1}{2} \qquad \text{and} \qquad x_{n+1} = \frac{1}{3}x_n + \frac{2}{3}.$$
Show that this sequence is increasing and bounded above by 1.

   (c) Using the continuum property of the real numbers, prove that every increasing sequence which is bounded above converges.

3. (a) State the *sandwich theorem* for sequences.

   (b) Compute the limit of the sequence
$$x_n = \sqrt[n]{3^n + 5^n + 7^n}$$
and justify your answer. *Hint: you may use (without proof) the fact that* $\lim_{n\to\infty} \sqrt[n]{a} = 1$ *for any constant* $a > 0$.

   (c) State the Intermediate Value Theorem for functions.

   (d) Prove that if $f : [0,2] \to [0,2]$ is any continuous function on $[0,2]$, then there exists a number $\xi \in [0,2]$ such that $f(\xi) = 2 - \xi$.

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4. (a) State the definition of convergence for a series $\sum_{n=1}^{\infty} a_n$.

   (b) Prove that if $a_n > 0$ for all n and $a_{n+1}/a_n \to \ell$ as $n \to \infty$ for some real number $\ell > 1$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

   For parts (c) and (d), you may use any techniques at your disposal.

   (c) Prove that for any $x > 1$, the series $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ diverges.

   (d) Prove that for all $x \in \mathbb{R}$, the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges absolutely.

5. (a) State the definitions of $\lim_{x \to a^+} f(x) = \xi$ and $\lim_{x \to a^-} f(x) = \xi$.

   (b) Consider the function

$$f(x) = \begin{cases} (x-1)^2 + 3 & \text{if } x \leq 2, \\ 4x - 2 & \text{if } x > 2. \end{cases}$$

   Using only the definitions, i.e. using ϵ and δ, show that

$$\lim_{x \to 2^-} f(x) = 4 \quad \text{and} \quad \lim_{x \to 2^+} f(x) = 6.$$

   Is f continuous at $x = 2$?

   (c) Find a sequence $\langle x_n \rangle$ such that $x_n \to 2$ but the sequence $\langle f(x_n) \rangle$ does not converge.

6. (a) State the Bolzano-Weierstrass theorem.

   (b) Let a and b be real numbers with $a < b$. Prove that every continuous function $f : [a, b] \to \mathbb{R}$ is bounded.

   (c) Consider the function

$$f(x) = \begin{cases} x^3 & \text{if } 0 \leq x < 1, \\ 0 & \text{if } x < 0 \text{ or } x \geq 1. \end{cases}$$

   Determine (with explanation) whether the function is bounded on each of the intervals $[-1, 1]$ and $\mathbb{R}$. Can you apply the theorem of part (b) to this function on either interval?

   (d) Determine (with explanation) whether the function f of part (c) achieves its maximum and minimum on each of the intervals $[-1, 1]$ and $\mathbb{R}$.

END OF PAPER